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Optimal Design of Stiffened Laminated Plates Using a Homotopy Method

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The use of a homotopy method is demonstrated for optimal design of a stiffened laminated plate for maximum buckling load. Instead of obtaining a single optimum, the homotopy technique generates in a single computer execution an entire family of optimum designs with a given parameter. In the present application the parameter is set to the total structural weight, and the optimal designs are obtained as a function of the weight of the laminated plates. It is seen that the number of simultaneous buckling modes of optimum plates is increased as the total weight is increased. So for low weights the optimal design starts with unimodal design and for higher weight the optimal design becomes bimodal, trimodal, and finally it becomes tetramodal.

Key Words: Optimization, Laminated Plates, Homotopy

1. Introduction

Conventionally, structural optimization is performed by either mathematical programming methods or optimality criteria methods. Both type of methods are based on iterative resizing of structures in the expectation that it will lead to the satisfaction of optimality conditions in the end of the optimization process. An alternative approach to find the optimal design is to treat the optimality conditions as a set of nonlinear equations and to solve them directly. This approach was rarely used in the past, first beacuse of the difficulty in dealing with such a large system of equations, and second because of multiplicity of solutions (mostly nonoptimal). Recent developments (Dennice and Schnabel, 1983) in methods for solving nonlinear equations make the direct solution of the optimality conditions more attractive. The difficulty arising from the multiple solutions is due to highly nonlinear nature of optimality conditions, however, the second order optimality check was shown to work effectively to identify the correct solutions (Shin, Haftka and Plaut, 1988a). Still another difficulty is that Newton-type methods, typically used to solve the nonlinear optimality conditions, are not guaranteed to converge unless the initial estimate is very close to the solution. A tracing technique was developed to eliminate this difficulty of convergence (Shin et al., 1988b, 1989). The tracing technique employs a homotopy method to locate the optimal solution with guaranteed convergence. The theory of globally convergent homotopy methods which are used in this study was developed in 1976 (Watson, 1979a, 1979b) and has been applied to solve a number of engineering problems. The full context of the theory and its applications are well documented in reference (Watson, 1986).

In this paper the use of the homotopy method is demonstrated for optimal design of a stiffened laminated palte for maximum buckling load. In many design problems with buckling load constraints, the optimal solutions are known to have repeated eigenvalues as their lowest buckling loads. The homotopy method in references (Shin et al., 1988b, 1989) is based on two different formulations that are applied separately to find the unimodal and bimodal solutions. This paper proposed a new procedure for handling problems when there are more than two buckling modes associated with the optimal design and the proce-

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dure is applied to find optimal designs of a stiffened laminated plate for different values of the total plate weight.

2. Stiffened Composite Plate

The plate considered in this study is square and simply supported along all four edges, as shown in Fig. 1. The plate has a symmetric (top and bottom) blade stiffener at center and is subjected to in-plane loads in the X-direction. Two forms of buckling of the stiffened plate are considered : the plate and stiffener, as one unit, can buckle in an overall buckling mode ; and the blade stiffener in itself can undergo local buckling. The buckling analyses of these two are performed separately. For the case of the overall plate buckling, the blade stiffener is treated as a beam reinforcing the plate. A finite element formulation is used, and the stiffness matrix and geometric stiffness matrix of the beam are calculated and combined with the corresponding matrices of the base plate. For the case of local stiffener buckling, the blade stiffener is treated as an independent plate subject to an in-plane compressive load. Assuming that three sides are simply supported and the fourth side is free we can obtain the local buckling load of the stiffener analytically.

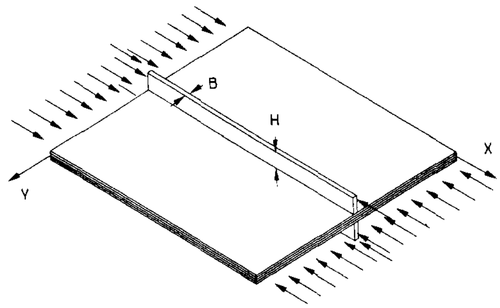

Fig. 1 Stiffened plate under inplane load

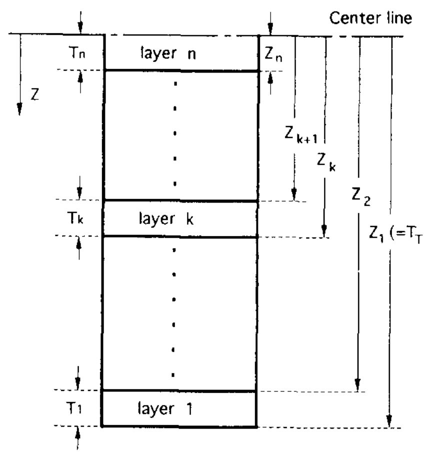

Fig. 2 Geometry of half of a 2n-layered symmetric laminate

2.1 Overall plate buckling analysis

As the stiffener is treated as a beam, the governing differential equation is

$$(E_b I_b W'')'' - P_b W'' = 0, \tag{1}$$

where W is the transverse deflection of the beam, a prime($'$) denotes differentiation with respect to the longitudinal coordinate X, and P_b is the beam axial load. Young's modulus of the beam in longitudinal direction is denoted by E_b, and the second moment of inertia of the beam is denoted by I_b and can be expressed as follows :

$$I_b = \frac{2}{3} B\{(H + T_T)^3 - T_T^3\}, \tag{2}$$

where B is the width, H is the height of the stiffener and T_T is half the total thickness of the laminate plate on which the stiffener is placed (see Figs. 1 and 2). The factor of 2 in the expression for I_b accounts for the two stiffeners, one on top, and one on bottom of the plate. Each of the geometric quantities and the material properties can be normalized dividing by the span length, L, or the layer stiffness of plate in fiber direction, E_{11}, as follows :

$$b = \frac{B}{L},\ h = \frac{H}{L},\ t_T = \frac{T_T}{L},$$
$$i_b = \frac{I_b}{L^4} = \frac{2}{3} b\{(h + t_T)^3 - t_T^3\},$$
$$e_b = \frac{E_b}{E_{11}},\ w = \frac{W}{L},\ x = \frac{X}{L},$$
$$p_b = \frac{1}{E_{11} L^2} P_b. \tag{3}$$

Then, we get the differential Eq. (1) also in nondimensional form :

$$(e_b i_b w'')'' + p_b w'' = 0. \tag{4}$$

The finite element discretization of the above beam equation leads to

$$[K]_b\{U\}_b - p_b[K_G]_b\{U\}_b = 0, \quad (5)$$

where $[K]_b$ is the global beam stiffness matrix, $[K_G]_b$ is the global beam geometric stiffness matrix, and $\{U\}_b$ is the generalized global displacement vector of the beam which has 2 degree-of-freedom$\{w, \frac{dw}{dx}\}$ at each node.

For the plate buckling analysis a 16 degree-of-freedom plate finite element (Yang, 1986) is used yielding the equation,

$$[K]\{U\} - n_x[K_G]\{U\} = 0, \quad (6)$$

where $[K]$ is the global plate stiffness matrix, $[K_G]$ is the global plate geometric stiffness matrix, and $\{U\}$ is the generalized global displacement vector of the plate which has 4 degree-of-freedom $\{w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x \partial y}\}$ at each node.

Since the terms w and $\frac{dw}{dx}$ in the vector $\{U\}_b$ in Eq. (5) are equivalent to the terms w and $\frac{\partial w}{\partial x}$ in the vector $\{U\}$ in Eq. (6), these two equations can be combined to get one matrix equation for overall buckling of the plate and stffener. The overall equation is

$$[K]\{U\} + [\underline{K}]_b\{U\} - n_x[K_G]\{U\} - p_b[\underline{K_G}]_b\{U\} = 0 \quad (7)$$

where $[K]$ and $[\underline{K}]_b$ are the plate and the beam stiffness matrix, respectively, $[K_G]$ and $[\underline{K_G}]_b$ are the plate and the beam geometric stiffness matrix, respectively. The matrices $[\underline{K}]_b$ and $[\underline{K_G}]_b$ are transformed from the matrices $[K]_b$ and $[K_G]_b$ to fit into the generalized displacement vector of plate $\{U\}$. The load carried by the beam is p_b and the load carried by the plate is n_x. These loads are proportional to the extensional stiffness of each. So the total load P on the stiffened plate is distributed as $pg1$ to the plate and $pg2$ to the stiffener, where $g1$ and $g2$ are expressed as follows:

$$g1 = \frac{e_p s_p}{e_p s_p + e_b s_b},$$

$$g2 = \frac{e_b s_b}{e_p s_p + e_b s_b}, \quad (8)$$

where s_p and s_b are the nondimensional cross-sectional area of plate and beam, and e_p and e_b are the nondimensional elastic moduli of the laminated plate and the beam. The expression for e_p are obtained from

$$\frac{1}{e_p} = \frac{a_{22}a_{66} - a_{26}^2}{\|[a]\|} t_T, \quad (9)$$

where $[a]$ is the nondimensional stretching stiffness matrix expressed as (see Fig. 2)

$$[a] = 2\sum_{k=1}^{n} [\bar{q}]k(z_k - z_{k+1}). \quad (10)$$

In above equation, n is half the number of total layers in plate, $\bar{q}$ is the nondimensional reduced stiffness of the laminate, and z_k is the nondimensional distance of each layer measured from the center of the laminate. Equation (7) may be written now as

$$[K]\{U\} + [\underline{K}]_b\{U\} - p(g1[K_G]\{U\} - g2[\underline{K_G}]_b\{U\}) = 0. \quad (11)$$

Now the overall buckling equation can be written in the form,

$$[K]_T\{U\} - p[K_G]_T\{U\} = 0, \quad (12)$$

where $[K]_T$ is the total stiffness of the plate and stiffener combination and $[K_G]_T$ is the total geometric stiffness of the plate and stiffener combination. This generalized eigenvalue problem is solved using the DNLASO subroutine from the package LASO2 (Scott and Parlett, 1983), which computes a few eigenvalues and associated eigenvectors of a large (sparse) symmetric matrix using the Lanczos algorithm (Golub, Underwood and Wilkinson, 1972).

2.2 Stiffener local buckling analysis

The local buckling of the stiffener is analyzed separately from the overall plate analysis. The stiffener is treated as a plate which is simply supported at 3 edges ($X=0$, $X=L$, $Z=0$) and is free at one edge ($Z=H$). It is subject to axial compressive load P_s. Because two opposite sides are simply supported, this problem can be solved analytically. The governing partial differential equation for the orthotropic plate buckling problem is given from (Jones, 1975) as

$$D_{11}\frac{\partial^4 V}{\partial X^4} + 2(D_{12} + 2D_{66})\frac{\partial^4 V}{\partial X^2 \partial Z^2} + D_{22}\frac{\partial^4 V}{\partial Z^4} - P_s\frac{\partial^2 V}{\partial Z^2} = 0, \quad (13)$$

where V is the plate deflection, D_{ij} are the components of the bending stiffness matrix, and P_s refers to the stiffener local buckling load. By applying the boundary conditions and assuming a Levy's solution of the form,

$$V=\sum_{m=1,3,5\cdots}^{\infty} F(Z)_m \sin\frac{m\pi X}{L}, \tag{14}$$

the buckling load P_s can be solved directly and the nondimensional buckling load p_s is obtained from the relation

$$p_s=\frac{1}{E_{11}L}P_s. \tag{15}$$

3. Optimization Problem

The optimization problem is to maximize the buckling load of the blade stiffened plate for a given total material volume (which is proportional to the total weight). The design variables are set as the nondimensional thickness of the individual lamina, t_i, and the nondimensional width and height of the stiffener. Here the nondimensional width, b, and the nondimensional height, h, of the stiffener are denoted as t_{n+1} and t_{n+2}, respectively. The nondimensional design variables, t_i, are subject to side constraints of

$$t_i \min \leq t_i \leq t_i \max \quad \text{for } i=1, 2, \cdots, (n+2), \tag{16}$$

where t_i max and t_i min are upper and lower bounds on the design variables, respectively, and n is half the number of layers for the symmetric laminate.

The optimization problem for maximizing the buckling load is written as

$$\begin{aligned} &\max_{\beta,\, t_i} \beta, \\ &\text{such that } p1 \geq \beta, \\ &\quad 0.999p2 \geq \beta, \\ &\quad 0.998p3 \geq \beta, \\ &\quad p_s \geq \beta, \\ &\quad 2\sum_{i=1}^{n} t_i + 2t_{n+1}t_{n+2} - \theta = 0, \\ &\text{and } t_i \min \leq t_i \leq t_i \max \text{ for } i=1, 2, \cdots, (n+2), \end{aligned} \tag{17}$$

where $p1$, $p2$, and $p3$ are the first three overall buckling loads of the stiffened plate which are obtained from overall FEM analysis, p_s is the local buckling load of the stiffener obtained analytically, and θ is the nondimensional total plate volume. The variable β is introduced to avoid having to maximize the minimum of $p1$, $p2$, $p3$, and p_s which is not a smooth function. The coefficients of 0.999, and 0.998 in the second and third constraints, are necessary to keep the eigenvalues $p1$, $p2$, and $p3$ distinct when the buckling mode is bimodal or trimodal, and allow the calculation of derivatives of these buckling loads.

A typical optimization method, applied to solve this problem, starts from a given design and continuously searches for better designs until it finds one optimum design. The intermediate designs along the path are of no interest to the structural designer. Here, instead, we use a method which traces an entire one-parameter family of optimal designs without going through any intermediate nonoptimal designs. For this we employ the homotopy method, a technique which has been used widely to solve nonlinear systems of equations. The basic idea of the method is to convert the system of equations into a set of ordinary differential equations with a parameter, called a homotopy parameter. Under certain assumptions, the method is guaranteed to converge to a solution even for highly nonlinear problems for which Newton-type iteration methods fail. The variable θ is chosen as the homotopy parameter, and for the initial conditions of the initial value problem we use the minimum-thickness plate in which all design variables at their lower bound. A sequence of optima corresponding to varying θ form the solution trajectory of the initial value problem. This use of the homotopy method for tracing optima was demonstrated in references (Shin et al., 1988b, 1989) only with equality constraints. Here the method is used with the formulation in which inequality constraints can be inclueded.

The equations defining the path of optimal designs are obtained using Lagrange multipliers, and are solved by the homotopy method. The optimum path consits of several smooth segments, with breaks in smoothness at points where the active constraint set changes. Changes in active constraints are associated with inequality constraints (here bound constraints on each of design variables) whjich may become active or inactive along the path. Along each segment, the active inequality constraints are treated as equality constraints,

$$t_j = t_j \text{ min or } t_j = t_j \text{ max} \quad \text{for } j \in I_A, \quad (18)$$

where I_A is the set of indices of design variables which are at a lower or upper bound. These variables are eliminated from the optimization problem, while the other variables are left unconstrained. The optimization problem along a segment can, therefore, be written as

$$\max_{\beta,\ t_i} \beta \quad \text{for } i \notin I_A,$$
$$\text{such that} \quad p1 \geq \beta,$$
$$0.999p2 \geq \beta,$$
$$0.998p3 \geq \beta,$$
$$p_s \geq \beta,$$
$$\text{and} \quad 2\sum_{i=1}^{n} t_i + 2t_{n+1}t_{n+2} - = 0. \quad (19)$$

The solution of the above problem requires dealing with three related problems : (1) solving the optimization problem along a segment, (2) locating the end of the segment where the set I_A changes, and (3) finding the set I_A for the next segment.

3.1 Stationary conditions

Using the Lagrange multiplier technique, the Lagrangian function β^* is

$$\begin{aligned}\beta^* = \beta &- \gamma 1(\beta + r_1{}^2 - p1) \\ &- \gamma 2(\beta + r2^2 - 0.999p2) \\ &- \gamma 3(\beta + \gamma_3^2 - 0.998p_3) \\ &- \gamma 4(\beta + \gamma_4{}^2 - p_s) \\ &- \mu(2\sum_{i=1}^{n} t_i + 2t_{n+1}t_{n+2} - \theta),\end{aligned} \quad (20)$$

where $\gamma 1$, $\gamma 2$, $\gamma 3$, $\gamma 4$, and μ are Lagrange multipliers and r_1, r_2, r_3, and r_4 are slack variables. Taking the first derivatives of β^* with respect to all these variables and setting them equal to zero, we obtain the stationary conditions,

$$1 - \gamma 1 - \gamma 2 - \gamma 3 - \gamma 4 = 0,$$
$$\gamma 1 \frac{\partial p_1}{\partial t_i} + 0.999\ \gamma 2 \frac{\partial p_2}{\partial t_i} + 0.998\ \gamma 3 \frac{\partial p_3}{\partial t_i} + \gamma 4 \frac{\partial p_s}{\partial t_i}$$
$$- c_{i\mu} = 0 \text{ for } i \notin I_A.$$
$$\gamma 1(p_1 - \beta) = 0,$$
$$\gamma 2(0.999p_2 - \beta) = 0,$$
$$\gamma 3(0.998p_3 - \beta) = 0,$$
$$\gamma 4(p_s - \beta) = 0,$$
$$2\sum_{i=1}^{n} t_i + 2t_{n+1}t_{n+2} - \theta = 0, \quad (21)$$

where c_i are the components of a coefficient vector, $\{2, 2, \cdots, 2, 2t_{n+2}, 2t_{n+1}\}^T$. These equations form a system of nonlinear equations to be solved for optimal design. The homotopy method is used to find the solution of these equations for varying θ.

3.2 Locating transition points

There are four types of transitions :

Type 1 : a bound constraint becoming active (i. e., being satisfied as an equality),

Type 2 : a bound constraint becoming inactive,

Type 3 : an inequality buckling load constraint becoming active,

Type 4 : an inequality buckling load constraint becoming inactive.

Transition points of type 1 ar located by checking the bound constraints (16). Transition points of type 2 are detected by checking positiveness of all Lagrange multipliers for bound constraints. These multipliers are given by

$$\gamma_{1i} = -\gamma 1 \frac{\partial p_1}{\partial t_i} - 0.999\ \gamma 2 \frac{\partial p_2}{\partial t_i} - 0.998\ \gamma 3 \frac{\partial p_3}{\partial t_i} - \gamma 4 \frac{\partial p_b}{\partial t_i} + c_{i\mu} \text{ for } t_i = t_{\min},$$
$$\lambda_{2i} = \gamma 1 \frac{\partial p_1}{\partial t_i} + 0.999\ \gamma 2 \frac{\partial p_2}{\partial t_i} + 0.998\ \gamma 3 \frac{\partial p_3}{\partial t_i} + \gamma 4 \frac{\partial p_b}{\partial t_i} - c_{i\mu} \text{ for } t_i = t_{\max}. \quad (22)$$

A transition of type 3 is detected by checking the buckling load constraints ;

$$p_1 \geq \beta,$$
$$0.999\ p2 \geq \beta,$$
$$0.998\ p3 \geq \beta,$$
$$p_s \geq \beta, \quad (23)$$

A transition of type 4 is detected by checking if the Lagrange multipliers associated with the buckling load constraints are positive ;

$$\gamma_i \geq 0, \quad \text{for } i = 1, 2, 3, 4. \quad (24)$$

At a transition point there are a number of solution paths which satisfy the stationary equations, so we need to choose a path which satisies the optimality conditions. Following discussion explains this procedure.

3.3 Choosing an optimum path

Once a transition point is located, we need to choose a path which satisfies the optimality conditions. Choosing an optimum path constitutes finding a set of active bound constraints for type 1 and type 2 transitions and the correct buckling modes for type 3 and type 4 transitions. These are obtained by using the Lagrange multipliers of the previous path and the sensitivity calculation on the buckling load. The procedure is explained separately for each type of transition.

A type 1 transition occurs when one of design variables, t_i, hits the upper or lower bound. Then t_i is set at t_i max or t_i min and treated as a constant value. The number of design variables is reduced by one.

At a type 2 transition, one of the Lagrange multipliers for the bound constraints, λ_{1i} and λ_{2i}, is found to be negative. The bound constraint corresponding to the negative λ_{1i} or λ_{2i} is set to be inactive and the number of design variables is increased by one.

A transition of type 3 is located by a buckling load p that becomes smaller tha β. Then the constraint for the buckling load is treated as an active constraint.

At a type 4 transition, One of the Lagrange multipliers for the buckling load constraints, γ_i, is known to be negative from the previous transition check, so the constraint corresponding to the negative γ_i is treated as inactive.

After we find the active design variables t and the active buckling load constraints, we can set the value of Lagrange multipliers, γ, for inactive constraints as zero. Then we only need to determine the values of active Lagrange multipliers γ, and μ at the transition point to complete the set of starting values for the next solution path. These are obtained by solving the stationary conditions (21) for given t which is a system of linear equations for these variables.

4. Results

A graphite/epoxy composite plate is selected whose material properties are given as follow : $E_{11}=21.374\times 10^{10}p_a(31.0\times 10^6 p_{si})$, $E_{22}=2.33\times 10^{10}$ $p_a(3.4\times 10^6 p_{si})$, $G_{12}=0.517\times 10^{10}p_a(0.75\times 10^6$ $p_{si})$, $\nu_{12}=0.28$.

Optimization results are shown for $(0°/90°)_S$ symmetric laminate with a blade-stiffener made from 0° material. Design variables include the ply thicknesses, the stiffener height, and the stiffener width. This laminate consists of four layers : however, only two of them are treated as design variables due to symmetry. The nondimensional minimum gage, $t_{\min}$, is set at 0.001 for all the design variables. The nondimensional total volume of the plate are expressen as $\theta=2(t_1+t_2+bh)$. And the design starts from $\theta=0.004002$ where all design variables are at the minimum gage.

Figure 3 shows the nondimensional value of layer thicknesses, t_1 and t_2, and the nondimensional width and the height, b and h, of the optimal design obtained for $0.004002\leq\theta\leq 0.04$.

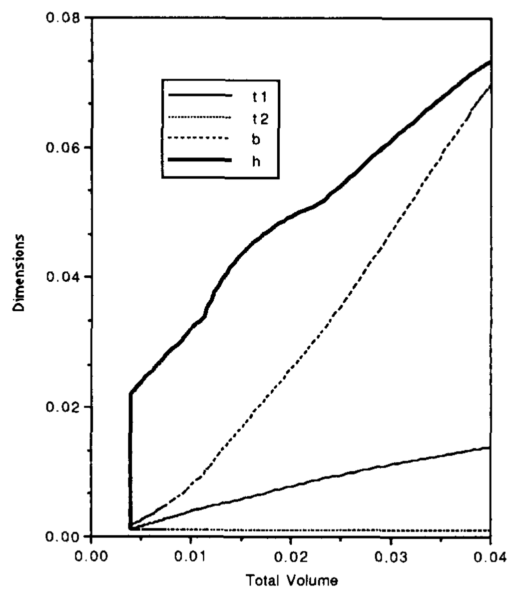

Fig. 3 Optimal dimensions of the plate in the total range

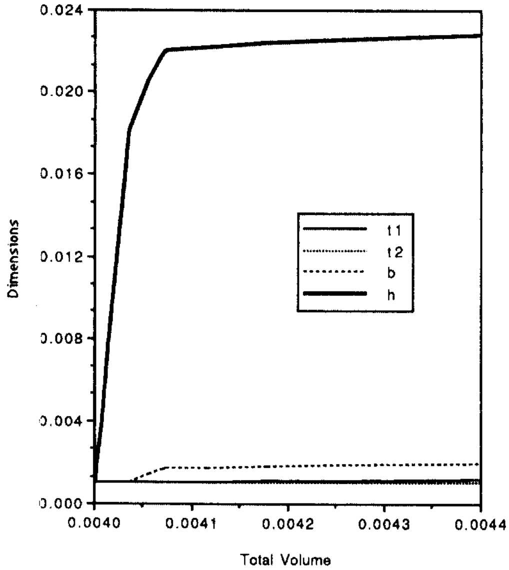

Fig. 4 Optimal dimensions of the plate in the initial range

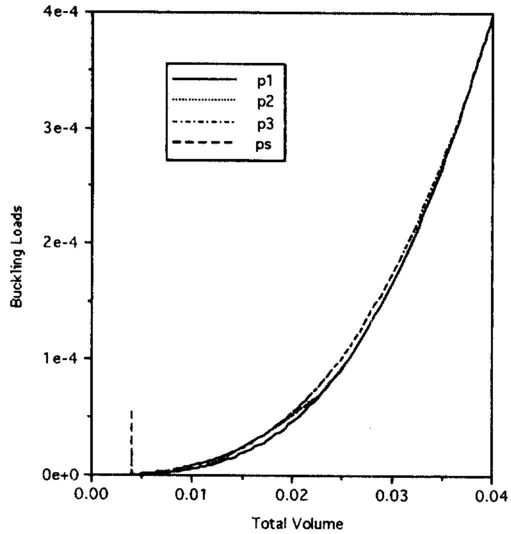

Fig. 5 Nondimensional buckling loads in the total range

Figure 4 is a blowup of this in the region of 0.004002 $\leq \theta \leq$ 0.0044. In Fig. 5, the first three nondimensional buckling loads, p_1, p_2, p_3 and the stiffener buckling load, p_s, of the optimal design are shown for 0.004002 $\leq \theta \leq$ 0.04 and its blowup in the range of 0.004002 $\leq \theta \leq$ 0.0044 is shown in Fig. 6.

Initially the optimum design starts with a unimodal design. The constraint for the first buckling load of the plate structure, p_1, is only active. The optimal design changes only the height of the stiffener while the other 3 design variables are at the minimum gage. When the total volume reaches at 0.004036, the constraint for the stiffener local buckling load, p_s, also becomes active, so that the design becomes bimodal. The width and the height of the stiffener change at the same time from this point. It is noticed that, up to this point, the total volume is increased only by 0.9% but the buckling load is increased by 148.1%. This shows the efficiency of the stiffener in designing composite plate structures which are subject to in-plane loads.

As θ approaches 0.004073, optimal designs become trimodal, i. e., first two overall buckling loads and the local stiffener buckling load have

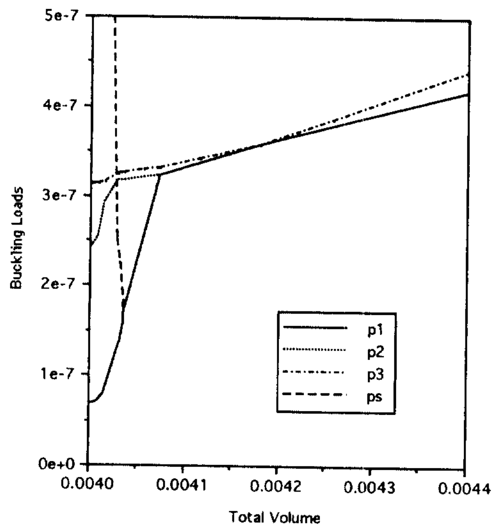

Fig. 6 Nondimensional buckling loads in the initial range

the same value. From this point on the minimum gage limit for t_1 which is the layer thickness corresponding to the 90° fibers, becomes inactive and t_1 starts to increase. Interestingly, it is seen that all 4 buckling loads merge together momentarily in the very short range of 0.00415 $\leq \theta \leq$ 0.00420. And for the range of 0.01132 $\leq \theta \leq$ 0.02440, the optimal designs become bimodal again.

After θ exceeds 0.03770 all 4 buckling load constraints become active and the buckling loads hold the same value (i. e., the 3 lowest eigenvalues and p_s have the same value). At this point all 4 minimum gage constraints become inactive. The overall buckling modes of the opimal design for $\theta=0.015$ are shown in Fig. 7 and for $\theta=0.04$ in Fig. 8.

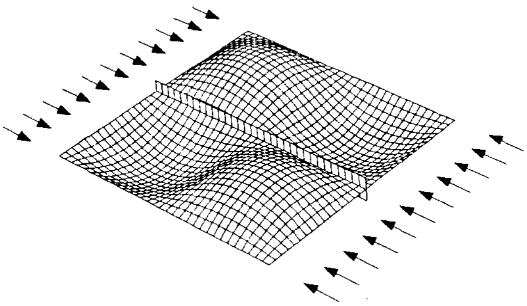

Fig. 7 Overall plate buckling mode of the optimal design ($\theta=0.015$)

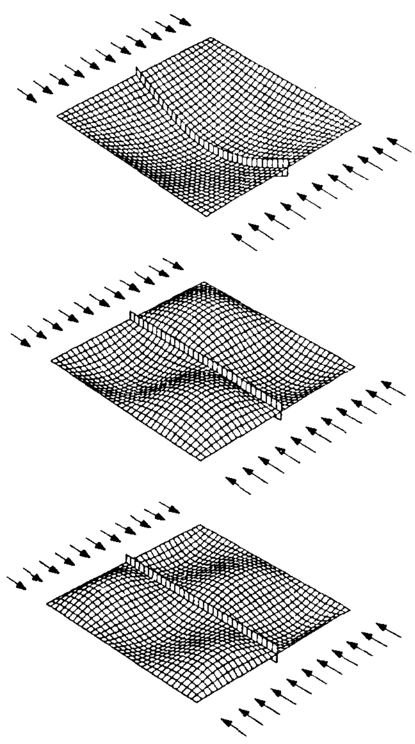

Fig. 8 Overall plate buckling modes of the optimal design ($\theta=0.04$)

5. Concluding Remarks

The use of the homotpy method is first motivated by the need to identify the optimum from other solutions of the optimality conditions. However, it has been found out to have a unique advatage over other conventional optimization techniques. Instead of obtaining a single optimum, which is typical of other methods, the homotopy technique generates in a single computer execution an entire family of optimum designs with a given parameter. This paper shows the technique can be implemented in a rater simple form for a problem having more than two buckling modes associated with the optimal design. Even though the paper discusses the use of homotopy technique in finding multiple optima parameterized by the total volume of a structure, it may be modified to solve a problem of a single optimum when the total structural volume is given. So, for future research, it is recommended to study the way how the technique cna be applied for a single optimum as well as its efficiency comparing to other search methods.

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